Definition: Σ is said to be locally l_p-constructible if there exist a function α of class K, a neighborhood V of zero in R^n, and a time $t_1 \geq 0$ such that for every $x_0 \in V$

$$\|g(\phi(.,0,x_0,0),0\|_p \geq \alpha(|\phi(t_1,0,x_0,0)|_p).$$

If $V = R^n$, then Σ is said to be globally l_p-constructible.

Remark 2: l_p-observability corresponds to the case $t_1 = 0$, and therefore it implies l_p-constructibility. The difference between these two notions is the following: roughly speaking, in the case of l_p-observability one is able to detect from the output $y = \{y(0), y(1), \cdots\}$ a nonzero initial state x_0 at time zero; in other words, this nonzero initial state can be detected using only present and future outputs. In the case of l_p-constructibility one is able to detect from the output y a nonzero state x_1 at time t_1; in other words, this state can be detected using a finite number of past outputs, present, and future ones. The error of Proposition 6)[1] regarding the l_p-observability of Σ_d is closely connected to this difference. It is proven that if $u = 0$ and x_h is any state at time h, one has $\|y\|_p \geq |x_h|_p$. In other words, the following result only has been proven [instead of 4)].

Proposition 1: Σ_d is globally l_p-constructible.

As was said above, in the case of a linear system, l_p-constructibility is equivalent to the usual constructibility:

Proposition 2: Consider the linear time-invariant system Σ_l defined by $x(t+1) = Ax(t)$, $y(t) = Cx(t)$. For any $p \in [1, \infty]$, the pair (C, A) is constructible (in the usual sense) if, and only if Σ_l is l_p-constructible.

Proof:

1) Assume that (C, A) is constructible, i.e.,

$$\mathrm{Ker}\ \Omega \subset \mathrm{Ker}\, A^n \tag{5}$$

where $\Omega = [C^T, A^T C^T, \cdots, (A^T)^{n-1} C^T]^T$. In other words, there exists a linear mapping $\Psi : R^{qn} \to R^n$ such that $A^n = \Psi\Omega$. Therefore, $x(n) = \Psi[y(0)^T, \cdots, y(n-1)^T]^T$, hence $|x(n)|_p \leq \|\Psi\| \|y\|_p$ and Σ_l is l_p-constructible.

2) Conversely, assume that (C, A) is not constructible. Then, there exists an initial state x_0 in $\mathrm{Ker}\ \Omega$ which does not belong to $\mathrm{Ker}\, A^n$. It follows that for any $k \geq 0$, x_0 cannot belong to $\mathrm{Ker}\, A^k$; hence for every $k \geq 0$, $x(k)$ is nonzero, whereas $y = 0$. Therefore, Σ_l is not l_p-constructible. $\square$

Let us now consider again the nonlinear system Σ. The following result is a refinement of Proposition 4.[1]

Proposition 3: Assume that Σ is locally l_p-reachable, locally l_p-constructible, and that the associated input–output operator is locally l_p-stable. Then, for any $p \in [1, \infty]$, zero is a stable equilibrium point (in the sense of Lyapunov) of the unforced system associated with Σ. This equilibrium point is locally asymptotically stable if $p < \infty$. This result still holds if "locally" is replaced everywhere by "globally."

Proof: The proof is identical to that of Proposition 4[1] up to the point where l_p-observability is used. Assuming that Σ is locally l_p-constructible, there exist a time t_1 and a function α of class K such that $\|y - y_{t-t_1-1}\|_p \geq \alpha(|x(t)|_p)$, $\forall t \geq \tau + t_1$, if the initial state x_0 at time τ is small enough. In addition, as the function f is continuous, the state $x(t)$ at times t such that $\tau \leq t \leq \tau + t_1 - 1$ are made arbitrarily small by taking x_0 small enough. The rest of the proof is identical to that of Proposition 1.[1] $\square$

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Simple Optimization Problems via Majorization Ordering

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***Abstract*—We introduce and explicitly solve a novel class of optimization problems which are motivated by load assignment issues in crossbar switches with output queueing. The optimization criterion is given in the majorization ordering sense. The solution to these problems indirectly provides solutions to a large class of convex optimization problems under a linear constraint.**

***Index Terms*— Convex optimization, crossbar switches, majorization, output queueing, Schur-convex mappings.**

I. Introduction

The notion of majorization (and its derivatives) provides a powerful tool to formalize statements concerning the relative size of the components of two vectors, viz., the components $(x_1, \cdots, x_K)$ of the vector $\mathbf{x}$ are "less spread out" than the components $(y_1, \cdots, y_K)$ of the vector $\mathbf{y}$. As elegantly demonstrated in the monograph of Marshall and Olkin [4], these notions have found widespread use in many diverse fields of mathematics and their applications. Recently, several authors have made use of majorization ideas to identify optimal scheduling and load balancing strategies for various resource allocation problems [1], [2]. In this paper, we consider a novel class of optimization problems which are motivated by load assignment issues in crossbar switches with output queueing; this application is discussed in some detail in Sections III and IV. These optimization problems are in the majorization sense and have the following generic form: for every vector $\mathbf{p}$ in $\mathbb{R}_+^K$ and every constant $c > 0$, we define the subset $\mathcal{A}(\mathbf{p}, c)$ of $[0,1]^K$ by

$$\mathcal{A}(\mathbf{p}; c) \equiv \left\{ \boldsymbol{\lambda} \in [0,1]^K : \sum_{k=1}^{K} p_k \lambda_k = c \right\}.$$

This set $\mathcal{A}(\mathbf{p}; c)$ is nonempty whenever c satisfies the condition

$$0 < c < \sum_{k=1}^{K} p_k. \tag{1}$$

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With $\prec$ denoting the majorization ordering (a precise definition is given at the end of this section), we seek vectors $\boldsymbol{\lambda}^+$ and $\boldsymbol{\lambda}^-$ in $\mathcal{A}(\mathbf{p}; c)$ such that

$$\gamma(\boldsymbol{\lambda}^-, \mathbf{p}) \prec \gamma(\boldsymbol{\lambda}, \mathbf{p}) \prec \gamma(\boldsymbol{\lambda}^+, \mathbf{p}), \qquad \boldsymbol{\lambda} \in \mathcal{A}(\mathbf{p}; c) \tag{2}$$

where we have used the notation

$$\gamma(\boldsymbol{\lambda}, \mathbf{p}) \equiv (\lambda_1 p_1, \cdots, \lambda_K p_K), \qquad \boldsymbol{\lambda} \in [0,1]^K, \qquad \mathbf{p} \in \mathbb{R}_+^K.$$

Explicit expressions for the optimizers $\boldsymbol{\lambda}^+$ and $\boldsymbol{\lambda}^-$ in terms of c and $\mathbf{p}$ are presented in Section II under the mild condition $\min_{k=1,\cdots,K} p_k > 0$. In addition to answering some natural questions raised by the results of [3], the material of Section II contains a curious byproduct which is of independent interest, especially to readers interested in convex optimization, and which we now develop.

When identifying $\boldsymbol{\lambda}^-$ [Theorem 2], we shall find it convenient to associate with $\mathcal{A}(\mathbf{p}; c)$ another subset $\mathcal{B}(\mathbf{p}; c) \equiv \{\gamma(\boldsymbol{\lambda}, \mathbf{p}),\ \boldsymbol{\lambda} \in \mathcal{A}(\mathbf{p}; c)\}$. The set $\mathcal{B}(\mathbf{p}; c)$ is also characterized by

$$\mathcal{B}(\mathbf{p}; c) \equiv \left\{ \mathbf{x} \in \mathbb{R}^K : 0 \le x_k \le p_k,\ k = 1, \cdots, K; \sum_{k=1}^K x_k = c \right\} \tag{3}$$

and is in one-to-one correspondence with $\mathcal{A}(\mathbf{p}; c)$ through the transformations

$$\mathbf{x} = \gamma(\boldsymbol{\lambda}, \mathbf{p}) \quad \text{if and only if} \qquad \lambda_i = \frac{x_i}{p_i}, \quad i = 1, \cdots, K. \tag{4}$$

The original problem of finding $\boldsymbol{\lambda}^-$ and $\boldsymbol{\lambda}^+$ in $\mathcal{A}(\mathbf{p}; c)$ satisfying (2) is equivalent to that of finding elements $\mathbf{x}^-$ and $\mathbf{x}^+$ in $\mathcal{B}(\mathbf{p}; c)$ such that

$$\mathbf{x}^- \prec \mathbf{x} \prec \mathbf{x}^+, \qquad \mathbf{x} \in \mathcal{B}(\mathbf{p}; c) \tag{5}$$

with $\mathbf{x}$ and $\boldsymbol{\lambda}$ related through (4).

The relation $\prec$ being a partial ordering on $\mathbb{R}^K$, it is natural to seek the mappings $\varphi : \mathbb{R}^K \to \mathbb{R}$ which are *monotonic* for the majorization ordering $\prec$, i.e., mappings with the property that $\varphi(\mathbf{x}) \le \varphi(\mathbf{y})$ whenever $\mathbf{x} \prec \mathbf{y}$. Such mappings are called Schur-convex mappings in honor of Schur who first studied them; the class of Schur-convex functions is very large and includes convex and symmetric mappings [4, C.2, p. 67] among other things. From (5) we conclude that

$$\varphi(\mathbf{x}^-) \le \varphi(\mathbf{x}) \le \varphi(\mathbf{x}^+), \qquad \mathbf{x} \in \mathcal{B}(\mathbf{p}; c) \tag{6}$$

for *any* Schur-convex function $\varphi : \mathbb{R}^K \to \mathbb{R}$, and the vector $\mathbf{x}^-$ (respectively, $\mathbf{x}^+$) is thus a solution to the optimization problem

$$\text{Minimize (respectively, Maximize) } \varphi \text{ over the set } \mathcal{B}(\mathbf{p}; c). \tag{7}$$

Note that the *same* vector $\mathbf{x}^-$ (respectively, $\mathbf{x}^+$) simultaneously solves *all* the corresponding problems (7) with Schur-convex functions φ.

Regarding the notation, the kth component of any element $\mathbf{x}$ in $\mathbb{R}^K$ is denoted either by x^k or by x_k, $k = 1, \cdots, K$, so that $\mathbf{x} \equiv (x^1, \cdots, x^K)$ or $(x_1, \cdots, x_K)$. A similar convention is used for $\mathbb{R}^K$-valued random variables (rv's). For any vector $\mathbf{x} = (x_1, \cdots, x_K)$ in $\mathbb{R}^K$, let $x_{(1)} \le x_{(2)} \le \cdots \le x_{(K)}$ denote the components of $\mathbf{x}$ arranged in increasing order. For vectors $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^K$, we say that $\mathbf{x}$ is *majorized* by $\mathbf{y}$, and write $\mathbf{x} \prec \mathbf{y}$, whenever the conditions

$$\sum_{i=1}^k x_{(i)} \ge \sum_{i=1}^k y_{(i)}, \qquad k = 1, 2, \cdots, K \tag{8}$$

and

$$\sum_{i=1}^K x_i = \sum_{i=1}^K y_i \tag{9}$$

hold. If conditions (8) all hold without (9), then we say that $\mathbf{x}$ is *weakly supermajorized* by $\mathbf{y}$ and write $\mathbf{x} \prec^w \mathbf{y}$ [4, A.2, p. 10].

II. The Main Results

In this section we establish the existence of and expressions for the vectors $\boldsymbol{\lambda}^-$ and $\boldsymbol{\lambda}^+$ satisfying (2). Throughout we assume the vector $\mathbf{p}$ to be selected so that

$$0 < p_1 \le p_2 \le \cdots \le p_K \tag{10}$$

and we leave it to the reader to check that there is no loss of generality in doing so.

Theorem 1: Assume (1) and (10) hold.

1) If $c \le p_K$, then the vector $\boldsymbol{\lambda}^+ \equiv (0, \cdots, 0, \frac{c}{p_K})$ is an element of $\mathcal{A}(\mathbf{p}; c)$ which satisfies

$$\gamma(\boldsymbol{\lambda}, \mathbf{p}) \prec \gamma(\boldsymbol{\lambda}^+, \mathbf{p}), \qquad \boldsymbol{\lambda} \in \mathcal{A}(\mathbf{p}; c). \tag{11}$$

2) If $p_K < c$, there exists a unique integer m $(m = 2, \cdots, K)$ such that

$$\sum_{k=m}^K p_k \le c < \sum_{k=m-1}^K p_k \tag{12}$$

and the vector $\boldsymbol{\lambda}^+$ defined by

$$\boldsymbol{\lambda}^+ \equiv \left(\underbrace{0, \cdots, 0}_{m-2}, \frac{c - \sum_{k=m}^K p_k}{p_{m-1}}, \underbrace{1, \cdots, 1}_{K-m+1} \right) \tag{13}$$

is an element of $\mathcal{A}(\mathbf{p}; c)$ which satisfies (11).

Proof—Claim 1: If $c < p_K$, then the vector $(0, \cdots, 0, \frac{c}{p_K})$ is indeed an element of $\mathcal{A}(\mathbf{p}; c)$, and the validity of (11) is well known in that case.

Claim 2: Set $Q_i \equiv \sum_{k=i}^K p_k$, $i = 1, \cdots, K$. Condition $p_K < c$ and (1) together imply $Q_K < c < Q_1$. The existence and uniqueness of the integer m satisfying (12) immediately follows from the strict monotonicity of $c - Q_i$, $i = 1, \cdots, K$, a fact implied by (10). We also conclude from (1) that $2 \le m \le K$ and $\boldsymbol{\lambda}^+$ is well defined. The definition (12) of m yields $0 \le \lambda_{m-1}^+ < 1$, and $\boldsymbol{\lambda}^+$ is therefore an element of $\mathcal{A}(\mathbf{p}; c)$. It is also plain that $\lambda_1^+ p_1 \le \lambda_2^+ p_2 \le \cdots \le \lambda_K^+ p_K$.

To establish (11) for some $\boldsymbol{\lambda}$ in $\mathcal{A}(\mathbf{p}; c)$, it suffices to show that for any permutation σ of $\{1, \cdots, K\}$, we have the inequalities

$$\sum_{i=k}^K \lambda_{\sigma(i)} p_{\sigma(i)} \le \sum_{i=k}^K \lambda_i^+ p_i, \qquad k = 1, \cdots, K. \tag{14}$$

Using (13) and the definition of $\mathcal{A}(\mathbf{p}; c)$, we see that

$$\sum_{i=k}^K \lambda_{\sigma(i)} p_{\sigma(i)} \le c = \sum_{i=k}^K \lambda_i^+ p_i, \qquad k = 1, \cdots, m-1$$

and (14) thus holds for $k = 1, \cdots, m-1$. On the other hand, because $\lambda_{\sigma(i)} \le \lambda_i^+ = 1$, $i = m, \cdots, K$, we have from (10) that

$$\sum_{i=k}^K \lambda_{\sigma(i)} p_{\sigma(i)} \le \sum_{i=k}^K p_{\sigma(i)} \le \sum_{i=k}^K \lambda_i^+ p_i, \qquad k = m, \cdots, K$$

and (14) also holds for $k = m, \cdots, K$. □

The quantities $D_0 \equiv K p_1 - c$ and

$$D_s \equiv (K - s) p_{s+1} - \left(c - \sum_{i=1}^s p_i \right), \qquad s = 1, \cdots, K-1$$

will be useful for characterizing the vector $\boldsymbol{\lambda}^-$ which satisfies (2).

Theorem 2: Assume (1) and (10) hold.

1) If $c \le Kp_1$, then the vector $\boldsymbol{\lambda}^- \equiv (\frac{c}{Kp_1}, \cdots, \frac{c}{Kp_K})$ is an element in $\mathcal{A}(\mathbf{p}; c)$ which satisfies

$$\gamma(\boldsymbol{\lambda}^-, \mathbf{p}) \prec \gamma(\boldsymbol{\lambda}, \mathbf{p}), \qquad \boldsymbol{\lambda} \in \mathcal{A}(\mathbf{p}; c). \tag{15}$$

2) If $Kp_1 < c$, then there exists an integer t $(t = 1, 2, \cdots, K-1)$ such that

$$D_{t-1} \le 0 \le D_t \tag{16}$$

and the vector $\boldsymbol{\lambda}^-$ defined by

$$\boldsymbol{\lambda}^- \equiv \left(\underbrace{1, \cdots, 1}_{t}, \frac{c - \sum_{k=1}^t p_k}{p_{t+1}(K-t)}, \cdots, \frac{c - \sum_{k=1}^t p_k}{p_K(K-t)} \right)$$

is an element of $\mathcal{A}(\mathbf{p}; c)$ which satisfies (15).

To clarify the proof of this result and to see why we might expect it in the first place, we turn to the change of variable (4) and focus on (5). We expect the minimizing element $\mathbf{x}^-$ to be as "balanced" as possible given the constraints defining $\mathcal{B}(\mathbf{p}; c)$; in fact, in the absence of the component constraints, the minimizing element would be simply given by $\frac{c}{K}(1, \cdots, 1)$. In general, this vector will not be the minimizing element since *a priori* it is possible for (1) to hold while $Kp_s < c$ for some $s = 1, \cdots, K-1$. This suggests that in constructing $\mathbf{x}^-$ we should attempt to keep as many components identical as possible while meeting the constraints on *all* the components. In view of (10) this construction would obviously start with $x_i^- = p_i$ for the smallest indexes and would lead to guessing $\mathbf{x}^-$ in the form

$$\mathbf{x}^- = (p_1, p_2, \cdots, p_s, a, \cdots, a) \tag{17}$$

for some integer $s = 1, \cdots, K-1$ [the case $s = K$ is ruled out by the strict inequality in (1)] and scalar $a > 0$. Such a choice (17) should be a reasonable candidate for the most balanced vector in $\mathcal{B}(\mathbf{p}; c)$, provided additional constraints are met. First, given s, we must have $p_s \le a$, for otherwise a more balanced vector in $\mathcal{B}(\mathbf{p}; c)$ could be constructed (by transfers [4, p. 134]) from $\mathbf{x}^-$ given by (17). The fact that $\mathbf{x}^-$ is an element of $\mathcal{B}(\mathbf{p}; c)$ further imposes $a \le p_{s+1}$ and $p_1 + \cdots + p_s + (K-s)a = c$. Hence, a is uniquely determined and the index s must be selected such that

$$p_s \le a \le p_{s+1} \quad \text{with} \quad a \equiv \frac{c - \sum_{k=1}^s p_k}{K-s}. \tag{18}$$

Note that (18) is equivalent to $D_{s-1} \le 0 \le D_s$, thereby giving a clue for the need of condition (16). In Theorem 3 below we show that the guess (17) and (18) is indeed correct.

Theorem 3: Assume (1) and (10) hold.

1) If $c \le Kp_1$, then the vector $\mathbf{x}^- \equiv \frac{c}{K}(1, \cdots, 1)$ is an element in $\mathcal{B}(\mathbf{p}; c)$ which satisfies

$$\mathbf{x}^- \prec \mathbf{x}, \qquad \mathbf{x} \in \mathcal{B}(\mathbf{p}; c). \tag{19}$$

2) If $Kp_1 < c$, then there exists an integer t $(t = 1, 2, \cdots, K-1)$ such that (16) holds, and the vector $\mathbf{x}^-$ defined by

$$\mathbf{x}^- \equiv \left(p_1, p_2, \cdots, p_t, \frac{c - \sum_{k=1}^t p_k}{K-t}, \cdots, \frac{c - \sum_{k=1}^t p_k}{K-t} \right) \tag{20}$$

is an element of $\mathcal{B}(\mathbf{p}; c)$ which satisfies (19).

Theorems 2 and 3 are clearly equivalent in view of the transformation (4).

Proof: We have $D_0 \le D_1 \le D_2 \le \cdots \le D_{K-1}$ as we note that

$$D_s - D_{s-1} = (K-s)(p_{s+1} - p_s) \ge 0, \qquad s = 1, \cdots, K. \tag{21}$$

Claim 1: If $c \le Kp_1$, then $\frac{c}{K} \le p_k$, $k = 1, \cdots, K$, and the vector $\frac{c}{K}(1, \cdots, 1)$ is indeed an element of $\mathcal{B}(\mathbf{p}; c)$; that it satisfies (19) is well known [4, p. 7].

Claim 2: The condition $Kp_1 < c$ is equivalent to $D_0 < 0$, and (1) yields $D_{K-1} > 0$. The existence of an integer t satisfying (16) follows from (21), and $\mathbf{x}^-$ given by (20) is thus well defined. As a consequence of (16) this vector is an element of $\mathcal{B}(\mathbf{p}; c)$, and its components are in *increasing* order. Thus, in order to establish its minimality within $\mathcal{B}(\mathbf{p}; c)$, we need only show for any element $\mathbf{x}$ of $\mathcal{B}(\mathbf{p}; c)$ that

$$\sum_{i=1}^k x_i^- \ge \sum_{i=1}^k x_{(i)}, \qquad k = 1, \cdots, K. \tag{22}$$

If $p_i < x_{(i)}$ for some $i = 1, \cdots, K$, then at most $(i-1)$ components of $\mathbf{x}$ do not exceed p_i, but this contradicts the fact that at least i components in $\mathbf{x}$ lie in the interval $[0, p_i]$. Hence, $x_{(i)} \le p_i$ for all $i = 1, \cdots, K$, and (22) holds for $k = 1, \cdots, t$.

Next, suppose that n is the first index greater than t for which (22) fails, i.e.,

$$\sum_{i=1}^n x_i^- < \sum_{i=1}^n x_{(i)} \quad \text{and} \quad \sum_{i=1}^{n-1} x_i^- \ge \sum_{i=1}^{n-1} x_{(i)}. \tag{23}$$

From (23) we note that

$$\sum_{i=1}^{n-1} x_i^- + x_n^- = \sum_{i=1}^n x_i^- < \sum_{i=1}^n x_{(i)} \le \sum_{i=1}^{n-1} x_i^- + x_{(n)} \tag{24}$$

so that $x_n^- < x_{(n)}$. On the other hand, the first part of (23) is equivalent to $\sum_{i=n+1}^K x_i^- > \sum_{i=n+1}^K x_{(i)}$ [via the equality constraint in (3)], and from (20) we get

$$(K-n)x_n^- = (K-n)\frac{c - \sum_{k=1}^t p_k}{K-t} > \sum_{i=n+1}^K x_{(i)} \ge (K-n)x_{(n)}.$$

The resulting inequality $x_{(n)} < x_n^-$ is in clear contradiction with the conclusion $x_n^- < x_{(n)}$ derived earlier from (24), and (22) must hold for $k = t+1, \cdots, K$. □

The integer t satisfying (16) may *not* be unique if some of the components of $\mathbf{p}$ are identical. However, in such circumstances, $\mathbf{x}^-$ defined through (20) is easily seen to be independent of the particular choice of t.

III. Nonblocking Switches with Output Queueing

In this section we present the model used by the authors in [3] to discuss various stochastic comparison results for a class of nonblocking switches with output queueing. With K input ports and L output ports, this model is parameterized by a vector of rates $\boldsymbol{\lambda}$ (in $[0,1]^L$) and by probability vectors $\mathbf{r}_k = (r_{k1}, \cdots, r_{kL})$ (in $\mathcal{S}_L \equiv \{\mathbf{r} = (r_1, \cdots, r_L) \in [0,1]^L : \sum_{\ell=1}^L r_\ell = 1\}$), $k = 1, \cdots, K$. We organize these K vectors into the $K \times L$ routing matrix $\mathbf{R} \equiv (r_{k\ell})$. With each set of such vectors, we associate $\{0,1\}$-valued rv's $\{A_{t+1}^k(\lambda_k),\ t = 0, 1, \cdots\}$ and $\{1, \cdots, L\}$-valued rv's $\{\nu_t^k(\mathbf{r}_k),\ t = 0, 1, \cdots\}$, $k = 1, \cdots, K$. These rv's are all defined on some probability triple $(\Omega, \mathcal{F}, \mathbf{P})$ and satisfy the following assumptions: 1) For each $k = 1, \cdots, K$, the rv's $\{A_{t+1}^k(\lambda_k),\ t = 0, 1, \cdots\}$ are i.i.d. rv's with

$$\mathbf{P}\left[A_{t+1}^k(\lambda_k) = 1\right] = 1 - \mathbf{P}\left[A_{t+1}^k = 0\right] = \lambda_k$$

for all $t = 0, 1, \cdots$; 2) For each $k = 1, \cdots, K$, the rv's $\{\nu_t^k(\mathbf{r}_k),\ t = 0, 1, \cdots\}$ are i.i.d. rv's with

$$\mathbf{P}\left[\nu_t^k(\mathbf{r}_k) = \ell\right] = r_{k\ell}, \qquad \ell = 1, \cdots, L$$

for all $t = 0, 1, \cdots$; and 3) The $2K$ collections of rv's $\{A^k_{t+1}(\lambda_k),\ t = 0, 1, \cdots\}$ and $\{\nu^k_t(\mathbf{r}_k),\ t = 0, 1, \cdots\}$, $k = 1, \cdots, K$, are *mutually independent*.

These quantities are given the following interpretation [3]: At the beginning of time slot $[t, t+1)$, new cells arrive into the system, with the $A^k_{t+1}(\lambda_k)$ cell arriving at the kth input port, $k = 1, \cdots, K$. The destination $\nu^k_t(\mathbf{r}_k)$ of the cell arriving at the kth input port is declared upon arrival. All cells which arrive during a time slot and which are destined for a given output port are transported across the switch during that single time slot and put into the corresponding output buffer in *random* order. Thus, during time slot $[t, t+1)$, the cells destined for the ℓth output port, $\ell = 1, \cdots, L$, form a batch of size $\xi^\ell_{t+1}(\boldsymbol{\lambda}, \mathbf{R})$ given by

$$\xi^\ell_{t+1}(\boldsymbol{\lambda}, \mathbf{R}) \equiv \sum_{k=1}^{K} \mathbf{1}\left[\nu^k_t(\mathbf{r}_k) = \ell\right] A^k_{t+1}(\lambda_k)$$

and at most one of these $\xi^\ell_{t+1}(\boldsymbol{\lambda}, \mathbf{R})$ cells can be transmitted, or equivalently, served during time slot $[t, t+1)$. Let $Q^\ell_t(\boldsymbol{\lambda}, \mathbf{R})$ denote the number of cells present at the beginning of time slot $[t, t+1)$ in the ℓth output buffer. If we assume the system to be initially empty at time $t = 0$, then the queue size process evolves according to the recursion

$$Q^\ell_{t+1}(\boldsymbol{\lambda}, \mathbf{R}) = \left[Q^\ell_t(\boldsymbol{\lambda}, \mathbf{R}) - 1\right]^+ + \xi^\ell_{t+1}(\boldsymbol{\lambda}, \mathbf{R}), \qquad t = 0, 1, \cdots$$
$$Q^\ell_0(\boldsymbol{\lambda}, \mathbf{R}) = 0. \tag{25}$$

For each $\ell = 1, \cdots, L$ and $n = 1, 2, \cdots$, let $D^\ell_n(\boldsymbol{\lambda}, \mathbf{R})$ denote the delay of the nth cell to arrive at the ℓth output port, i.e., $D^\ell_n(\boldsymbol{\lambda}, \mathbf{R})$ represents the time that elapses between the arrival of the nth cell at the ℓth output port and the end of its transmission. At each of the output queues, we assume that batches are processed in the order of arrival, i.e., all cells in the mth batch are served before the cells in the $(m+1)$st batch, $m = 1, 2, \cdots$, but the order of service within a given batch is random. As a result, the delay process of the nth cell can be decomposed into two successive stages, so that $D^\ell_n(\boldsymbol{\lambda}, \mathbf{R}) = W^\ell_n(\boldsymbol{\lambda}, \mathbf{R}) + B^\ell_n(\boldsymbol{\lambda}, \mathbf{R})$, where the rv $W^\ell_n(\boldsymbol{\lambda}, \mathbf{R})$ counts the number of slots required for transmitting all the cells in the batches which have arrived before that containing the nth cell, and the rv $B^\ell_n(\boldsymbol{\lambda}, \mathbf{R})$ counts the number of slots that the nth cell needs to wait before it is served, once the batch to which it belongs starts being served.

The recursions (25) are very similar to the Lindley recursion for single server queues, and by arguments similar to those used in that context, the following facts can be shown: For each $\ell = 1, \cdots, L$, we define the offered load to the ℓth output buffer by

$$\rho_\ell(\boldsymbol{\lambda}, \mathbf{R}) \equiv \sum_{k=1}^{K} \lambda_k r_{k\ell}. \tag{26}$$

If $\rho_\ell(\boldsymbol{\lambda}, \mathbf{R}) < 1$, then there exists an $\mathbb{N}$-valued rv $Q^\ell(\boldsymbol{\lambda})$ such that the one-dimensional convergence $Q^\ell_t(\boldsymbol{\lambda}, \mathbf{R}) \Longrightarrow_t Q^\ell(\boldsymbol{\lambda}, \mathbf{R})$ takes place, and the ℓth output queue is then said to be *stable*. In that case, we also have $D^\ell_n(\boldsymbol{\lambda}, \mathbf{R}) \Longrightarrow_n D^\ell(\boldsymbol{\lambda}, \mathbf{R})$ for some rv's $D^\ell(\boldsymbol{\lambda}, \mathbf{R})$ given by $D^\ell(\boldsymbol{\lambda}, \mathbf{R}) =_{st} Q^\ell(\boldsymbol{\lambda}, \mathbf{R}) + B^\ell(\boldsymbol{\lambda}, \mathbf{R})$ where $B^\ell(\boldsymbol{\lambda}, \mathbf{R})$ is the forward recurrence time associated with $\xi^\ell_1(\boldsymbol{\lambda}, \mathbf{R})$, and $Q^\ell(\boldsymbol{\lambda}, \mathbf{R})$ and $B^\ell(\boldsymbol{\lambda}, \mathbf{R})$ are independent rv's.

IV. Comparison Results and One-Dimensional Bounds

We now present several stochastic comparison results that describe how changes in arrival rates and routing probabilities affect the various performance measures; these results were obtained in the companion paper [3]. To simplify the presentation, for each rate vector $\boldsymbol{\lambda}$ and routing matrix $\mathbf{R}$, we write

$$\gamma_\ell(\boldsymbol{\lambda}, \mathbf{R}) \equiv (\lambda_1 r_{1\ell}, \cdots, \lambda_K r_{K\ell}), \qquad \ell = 1, \cdots, L.$$

Throughout, the notation $\leq_{icx}$ is used to denote the convex increasing ordering on the collection of distributions on $\mathbb{R}$ [5].

Theorem 4: Assume that for some $\ell = 1, \cdots, L$, the comparison

$$\gamma_\ell(\boldsymbol{\lambda}, \mathbf{R}) \prec^w \gamma_\ell(\boldsymbol{\lambda}', \mathbf{R}') \tag{27}$$

holds. Then, we have $Q^\ell_t(\boldsymbol{\lambda}', \mathbf{R}') \leq_{icx} Q^\ell_t(\boldsymbol{\lambda}, \mathbf{R})$ for all $t = 0, 1, \cdots$. If in addition $\rho_\ell(\boldsymbol{\lambda}, \mathbf{R}) < 1$, then in statistical equilibrium we have $Q^\ell(\boldsymbol{\lambda}', \mathbf{R}') \leq_{icx} Q^\ell(\boldsymbol{\lambda}, \mathbf{R})$ and $D^\ell(\boldsymbol{\lambda}', \mathbf{R}') \leq_{icx} D^\ell(\boldsymbol{\lambda}, \mathbf{R})$.

Under (27), the stability condition $\rho_\ell(\boldsymbol{\lambda}, \mathbf{R}) < 1$ implies $\rho_\ell(\boldsymbol{\lambda}', \mathbf{R}') < 1$, so that the ℓth output queue is stable in both systems and the comparisons have a well-defined meaning. Furthermore, if the total load (26) to the ℓth output queue is constrained to some given value, then (27) is equivalent to

$$\gamma_\ell(\boldsymbol{\lambda}, \mathbf{R}) \prec \gamma_\ell(\boldsymbol{\lambda}', \mathbf{R}'). \tag{28}$$

Theorem 4 thus suggests a way to obtain lower and upper bounds on the queue size metrics (among other things) by seeking the "extremizers" in the conditions (28) under certain load constraints; this leads to the generic problems presented in Section I. For the remainder of the discussion, we fix some $\ell = 1, \cdots, L$ and consider two situations which are both associated with the ℓth output queue.

Problem A: For a given arrival vector $\boldsymbol{\lambda}$, we seek the routing matrix $\mathbf{R}$ which icx-minimizes (respectively, icx-maximizes) the performance measures at the ℓth output queue subject to the total load (26) to the ℓth output queue being constrained to some given value, say ρ_ℓ. In view of Theorem 4 (and remarks following it) it suffices to identify routing matrices $\mathbf{R}^-$ and $\mathbf{R}^+$ such that

$$\gamma_\ell(\boldsymbol{\lambda}, \mathbf{R}^-) \prec \gamma_\ell(\boldsymbol{\lambda}, \mathbf{R}) \prec \gamma_\ell(\boldsymbol{\lambda}, \mathbf{R}^+) \tag{29}$$

among the routing matrices $\mathbf{R}$ which satisfy the load equation

$$\sum_{k=1}^{K} \lambda_k r_{k\ell} = \rho_\ell. \tag{30}$$

Being concerned only with the ℓth output queue, we need only specify the ℓth column of the routing matrices involved, and the problem thus reduces to finding vectors $\mathbf{c}^-$ and $\mathbf{c}^+$ in the set $\mathcal{A}(\boldsymbol{\lambda}; \rho_\ell)$ such that $\gamma(\boldsymbol{\lambda}, \mathbf{c}^-) \prec \gamma(\boldsymbol{\lambda}, \mathbf{c}) \prec \gamma(\boldsymbol{\lambda}, \mathbf{c}^+)$ for all $\mathbf{c}$ in $\mathcal{A}(\lambda; \rho_\ell)$. With this notation, $\mathbf{c}^-$, $\mathbf{c}$, and $\mathbf{c}^+$ represent the ℓth column of the routing matrices $\mathbf{R}^-$, $\mathbf{R}$, and $\mathbf{R}^+$, respectively, appearing in (29). By invoking the results of Section II we can now easily characterize $\mathbf{c}^-$ and $\mathbf{c}^+$, and we do so under the assumptions $0 < \lambda_1 \leq \cdots \leq \lambda_K$ and $0 < \rho_\ell < \sum_{k=1}^{K} \lambda_k$.

By specializing Theorem 1, we find

$$\mathbf{c}^+ \equiv \left(\underbrace{0, \cdots, 0}_{m-2}, a^+, \underbrace{1, \cdots, 1}_{K-m+1}\right)$$

with m and a^+ given by

$$m \equiv \min\left\{i = 2, \cdots, K : \sum_{k=i}^{K} \lambda_k \leq \rho_\ell\right\}$$

and

$$a^+ \equiv \frac{\rho_\ell - \sum_{k=m}^{K} \lambda_k}{\lambda_{m-1}}.$$

Here we use the natural convention that if the set of indexes entering the definition of m is empty, then $m = K+1$ and $\sum_{k=m}^{K} \lambda_k = 0$. On the other hand, Theorem 2 immediately yields the following: if

$\rho_\ell \le K\lambda_1$, then $\mathbf{c}^- = (\frac{\rho_\ell}{K\lambda_1}, \cdots, \frac{\rho_\ell}{K\lambda_K})$, whereas if $K\lambda_1 < \rho_\ell$, then there exists t $(t = 1, \cdots, K-1)$ such that

$$\lambda_t \le a^- \equiv \frac{\rho_\ell - \sum_{k=1}^t \lambda_k}{K-t} \le \lambda_{t+1} \tag{31}$$

and we have

$$\mathbf{c}^- = \left(\underbrace{1, \cdots, 1}_{t}, \frac{a^-}{\lambda_{t+1}}, \cdots, \frac{a^-}{\lambda_K}\right). \tag{32}$$

In sum, for a given arrival vector $\boldsymbol{\lambda}$, any routing matrix whose ℓth column is given by $\mathbf{c}^+$ (respectively, $\mathbf{c}^-$) will icx-minimize (respectively, icx-maximize) the performance measures at the ℓth output queue subject to the load constraint (30).

When the input ports are *equiloaded*, i.e., $\boldsymbol{\lambda} \equiv \frac{\lambda}{K}(1, \cdots, 1)$ for some $\lambda > 0$, the feasibility constraint reads $\rho_\ell < \lambda$, and additional simplifications occur. As expected, we find that (31) and (32) specialize to $\mathbf{c}^- = \frac{\rho_\ell}{\lambda}(1, \cdots, 1)$, and we have

$$m = \left\lceil K\left(1 - \frac{\rho_\ell}{\lambda}\right)\right\rceil + 1 \quad \text{and} \quad a^+ = \frac{K\rho_\ell - (K-m+1)\lambda}{\lambda}.$$

Problem B: For a given routing matrix $\mathbf{R}$, we now seek the arrival vector $\boldsymbol{\lambda}$ which icx-minimizes (respectively, icx-maximizes) the performance measures at the ℓth output queue subject to the load constraint (30) at the ℓth output queue. Again, we need only identify arrival vectors $\boldsymbol{\lambda}^-$ and $\boldsymbol{\lambda}^+$ such that $\gamma_\ell(\boldsymbol{\lambda}^-, \mathbf{R}) \prec \gamma_\ell(\boldsymbol{\lambda}, \mathbf{R}) \prec \gamma_\ell(\boldsymbol{\lambda}^+, \mathbf{R})$ for all arrival vectors $\boldsymbol{\lambda}$ satisfying (30). With $\mathbf{c}_\ell$ denoting the ℓth column of the routing matrix $\mathbf{R}$, the problem reduces to finding vectors $\boldsymbol{\lambda}^-$ and $\boldsymbol{\lambda}^+$ in $\mathcal{A}(\mathbf{c}_\ell; \rho_\ell)$ such that

$$\gamma(\boldsymbol{\lambda}^-, \mathbf{c}_\ell) \prec \gamma(\boldsymbol{\lambda}, \mathbf{c}_\ell) \prec \gamma(\boldsymbol{\lambda}^+, \mathbf{c}_\ell), \qquad \boldsymbol{\lambda} \in \mathcal{A}(\mathbf{c}_\ell; \rho_\ell). \tag{33}$$

In order to characterize $\boldsymbol{\lambda}^-$ and $\boldsymbol{\lambda}^+$, we again invoke the results of Section II under the assumptions $0 < r_{1\ell} \le \cdots \le r_{K\ell}$ and $0 < \rho_\ell < \sum_{k=1}^K r_{k\ell}$. This time, under the same convention as before, we have

$$\boldsymbol{\lambda}^+ = \left(\underbrace{0, \cdots, 0}_{m-2}, b^+, \underbrace{1, \cdots, 1}_{K-m+1}\right)$$

with m and b^+ given by

$$m \equiv \min\left\{i = 2, \cdots, K : \sum_{k=i}^K r_{k\ell} \le \rho_\ell\right\}$$

and

$$b^+ \equiv \frac{\rho_\ell - \sum_{k=m}^K r_{k\ell}}{r_{m-1,\ell}}.$$

From Theorem 2, if $\rho_\ell \le K r_{1\ell}$, then $\boldsymbol{\lambda}^- = (\frac{\rho_\ell}{K r_{1\ell}}, \cdots, \frac{\rho_\ell}{K r_{K\ell}})$, whereas if $K r_{1\ell} < \rho_\ell$, then there exists t $(t = 1, \cdots, K-1)$ such that

$$r_{t\ell} \le b^- \equiv \frac{\rho_\ell - \sum_{k=1}^t r_{k\ell}}{K-t} \le r_{t+1,\ell}$$

and we have

$$\boldsymbol{\lambda}^- = \left(\underbrace{1, \cdots, 1}_{t}, \frac{b^-}{r_{t+1,\ell}}, \cdots, \frac{b^-}{r_{K\ell}}\right).$$

Therefore, for a given routing matrix $\mathbf{R}$, the arrival vector $\boldsymbol{\lambda}^+$ (respectively, $\boldsymbol{\lambda}^-$) icx-minimizes (respectively, icx-maximizes) the performance measures at the ℓth output queue subject to the load constraint (30) at that queue. In general, these one-dimensional results are not independent of ℓ, i.e., the vectors $\boldsymbol{\lambda}^-$ and $\boldsymbol{\lambda}^+$ do *not* simultaneously satisfy (33) under (30) for *all* $\ell = 1, \cdots, L$. This can be remedied by considering the often-studied situation where the addressing scheme is *input independent* in the sense that $\mathbf{R}$ has all its row identical with $\mathbf{r}_k = \mathbf{r}$, $k = 1, \cdots, K$, for some vector $\mathbf{r} = (r_1, \cdots, r_L)$ in $\mathcal{S}_L$. In that case, the constraint (30) becomes $\sum_{k=1}^K \lambda_k = \frac{\rho_\ell}{r_\ell} \equiv \lambda$, with $0 < \lambda < K$, and the inequalities (29) are satisfied simultaneously for *all* $\ell = 1, \cdots, L$ by the vectors $\boldsymbol{\lambda}^- = \frac{\lambda}{K}(1, \cdots, 1)$ and

$$\boldsymbol{\lambda}^+ = \left(\underbrace{0, \cdots, 0}_{m-2}, \lambda - (K-m+1), \underbrace{1, \cdots, 1}_{K-m+1}\right)$$

with

$$m \equiv \lceil K - \lambda \rceil + 1.$$

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Diagonal Matrix Solutions of a Discrete-Time Lyapunov Inequality

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***Abstract*—Diagonal solutions of a Lyapunov inequality for companion matrices are studied. Such solutions are required if states of a discrete-time linear system are computed with a finite-precision arithmetic.**

***Index Terms*—Companion matrix, critical exponent, diagonal stability, discrete Lyapunov equation.**

I. Introduction

Let

$$x(i+1) = Ax(i), \qquad x(0) = x_0 \tag{1}$$

be a discrete-time linear system with $x(i) = (x_1(i), \cdots, x_n(i))^T \in C^n$. It is well known that (1) is asymptotically stable if and only if there exists a matrix $P > 0$ (positive-definite) such that

$$A^*PA - P = -Q^*Q \tag{2}$$

and

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